Date:

Registration number:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.Sc. Mathematics - IV SEMESTER
SEMESTER EXAMINATION: APRIL 2022
(Examination conducted in July 2022)

MT 0118 – MEASURE AND INTEGRATION

Time- 2 ½ hrs Max Marks-70

This question paper contains 2 printed pages and one part.

Answer any 7 questions

1. (a) Let $\mathcal{A}$ be a σ-algebra. Prove that $\mathcal{A}$ is closed under countable intersection. (3)
   (b) State and prove "Excision Property". (4)
   (c) True or False: Any set of measure zero is countable. Justify. (3)

2. (a) When do we say a set E is measurable? Give an example of measurable set of finite measure. (2)
   (b) If $\{\mathrm{A_k}\}_{\mathrm{k}=1}^{\infty}$ is an ascending collection of measurable sets, then prove that (6)
$$\mathrm{m}\left(\bigcup_{\mathrm{k}=1}^{\infty}\mathrm{A_k}\right)=\lim_{\mathrm{k}\to\infty}\mathrm{m}(\mathrm{A_k})$$
   (c) State countable monotonicity property of Lebesgue Measure. (2)

3. (a) Prove that $f(x) = x$ on $[2,3]$ is measurable. (2)
   (b) Let f and g be measurable functions on E that are finite a.e. on E. Prove that for any α and β, $\alpha f + \beta g$ is measurable on E. (8)

4. (a) Define simple function. Consider the function $\psi(x) = \begin{cases} \lceil x \rceil & \text{if } x \in [0,3) \\ \lfloor x \rfloor & \text{if } x \in [3,5) \end{cases}$ defined on $E = [0,5)$. Give its canonical representation and also find $\int_E \psi$. (6)
   (b) State The Simple Function Approximation Lemma and Egoroff's Theorem. (4)

5. (a) Let f be bounded measurable functions on a set of finite measure E. Then for any α, prove that $\int_E \alpha f = \alpha \int_E f$. (4)
   (b) Find two sets of bounded measurable functions f, g on a set of finite measure E such that the identity $\int_E fg = \int_E f \cdot \int_E g$: i) is true ii) is disproved. Justify. (6)

6. (a) State and prove Bounded Convergence Theorem. (8)
   (b) State Integral Comparison Test. (2)

7. State and prove Fatou's Lemma and state Monotone Convergence Theorem. (10)

8. (a) When do we say a covering of a set is in the sense of Vitali? Check if the covering of $E = [1, 2]$ given by $\left\{\left[1-\frac{1}{n}, 2+\frac{1}{n}\right]\right\}_{n=1}^{\infty}$ and $\cup_{n=1}^{\infty}\left\{\left(x-\frac{1}{n}, x+\frac{1}{n}\right)\middle| x \in E\right\}$ are covering in the sense of Vitali, with proper justification. (6)

(b) State Vitali Covering Lemma. (2)

(c) Given that f is an increasing function on the closed, bounded interval $[a, b]$. Which of the following is/are true? (2)

i) For each $\alpha > 0, m^*\{x \in (a, b) | \overline{D}f(x) \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)]$

ii) $m^*\{x \in (a, b) | \overline{D}f(x) = \infty\} > 0$

iii) There exists a unique $\alpha > 0$ where $m^*\{x \in (a, b) | \overline{D}f(x) \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)]$

iv) $m^*\{x \in (a, b) | \overline{D}f(x) = \infty\} = 0$

9. (a) Define bounded variation and show that $f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$ is not of bounded variation on $[0,1]$. (5)

(b) When do we say that a function is absolutely continuous? Show that the function $\sin x$ is absolutely continuous on $[0, 2\pi]$ (3)

(c) Let the function f be absolutely continuous on the closed, bounded interval [a, b]. Then which of the following is/are true? (2)

i) f is differentiable a.e. on (a, b)

ii) f' is integrable over $[a, b]$

iii) $\int_a^b f = f'(a) - f'(b)$

iv) $\int_a^b f' = f(a) - f(b)$

10. (a) Define conjugate of a number $p \in (0, \infty)$. What is the conjugate of 1? Also find the conjugate of 3. (4)

(b) State Hölder's Inequality for $L^p(E)$. State the special case of Hölder's Inequality when $p = q = 2$ in $L^p(E)$. What is the name of that inequality? (4)

(c) Let X be a normed linear space. Then which of the following statements is/are true? (2)

i) Every Cauchy sequence is convergent.

ii) Every Convergent sequence is Cauchy.

iii) Every Cauchy sequence is rapidly Cauchy.

iv) Every Cauchy sequence has a rapidly Cauchy subsequence