Register Number:

Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.SC MATHEMATICS - IV SEMESTER
SEMESTER EXAMINATION: APRIL 2022
(Examination conducted in July 2022)

MT DE0618: Representation Theory of Finite Groups

Duration: 2.5 Hours **Max. marks:**70

1. The paper contains two pages.
2. Attempt any **SEVEN FULL** questions.
3. Each question carries 10 marks.

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1. Show that every representation of a **finite** group is equivalent to a unitary representation. Deduce that a representation of a finite group is either irreducible or decomposable. [**10 m**]

2. (a) Let G be a **finite** group and $\phi : G \to GL(V)$ be a representation of degree 3. Show that if there is no common eigenvector v to all ϕ_g with $g \in G$ then ϕ is irreducible . [**4 m**]
   (b) Show by an example that part (a) is false if we drop the finiteness condition. [**6 m**]

3. (a) State and prove Schur's Lemma. [**6 m**]
   (b) Let G be an abelian group. Show that any irreducible representation of G has degree one. [**4 m**]

4. (a) Let ϕ, ρ be irreducible representations of a finite group G. Show that

$$\langle \chi_\phi, \chi_\rho \rangle = \begin{cases} 1 & \text{if } \phi \sim \rho \\ 0 & \text{if } \phi \nsim \rho. \end{cases}$$

[**6 m**]

   (b) Let χ be a non trivial irreducible character of a finite group G. Show that $\sum_{g \in G} \chi(g) = 0$. [**4 m**]

5. (a) State and prove second orthogonality relation. [**6 m**]

(b) Let G be a group of order 12 which has exactly four conjugacy classes. Complete the character table. **[4 m]**

	$g_1 = e$	g_2	g_3	g_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4				

6. (a) Let G be a finite abelian group and $L(G) = \{f | f : G \to \mathbb{C}\}$. Show that $(L(G), +, *)$ is isomorphic to $(L(\widehat{G}), +, \cdot)$ as $\mathbb{C}$-algebras, where " $*$ " is the convolution product and " $\cdot$ " is the point-wise multiplication. **[6 m]**

   (b) Let G be an abelian group and $a \in L(G)$. Let $A : L(G) \to L(G)$ be the convolution operator defined by $A(b) = a * b$. Show that A is a linear transformation with χ as an eigenvector with eigenvalue $\widehat{a}(\chi)$ for all $\chi \in \widehat{G}$. **[4 m]**

7. Draw the Cayley graph of $\mathbb{Z}_6$ with respect to the set $S = \{\pm[2], \pm[3]\}$. Write down the adjacency matrix of the graph and find all the eigenvalues of it. Also write down the corresponding eigenvectors for the positive eigenvalues. **[10 m]**

8. (a) State and prove Dimension Theorem. **[6 m]**

   (b) Let G be a non-abelian group of order 39. Determine the degrees of irreducible representations of G and how many irreducible representations G has of each degree (up to equivalence). Determine the number of conjugacy classes of G. **[4 m]**

9. (a) Let $\sigma : G \to S_X$ be a group action. Let $\mathcal{O}_1, \cdots, \mathcal{O}_m$ be the orbits of G on X and define $v_i = \sum_{x \in \mathcal{O}_i} x$. Then show that $\{v_1, \cdots, v_m\}$ is a basis for $\mathbb{C}X^G$. **[6 m]**

   (b) State and Prove Burnside's Lemma. **[4 m]**

10. Compute the character table of S_4. **[10 m]**

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