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ST. JOSEPH'S COLLEGE (AUTONOMOUS), BANGALORE-27

M.Sc. Physics – SEMESTER I

MID-SEMESTER TEST - AUGUST 2016

PH 7215: MATHEMATICAL PHYSICS

Time: 1 hr. 30 min. **Maximum Marks: 35**

This question paper has 3 printed pages and 1 part

Answer any SEVEN questions. Each question carries 5 marks.

1.

(a) A complex number is represented in polar form as $z=re^{i\theta}$. If we multiply z by $e^{i\varphi}$ we have $z'=e^{i\varphi}z=re^{i(\theta+\varphi)}$. Since $z=x+iy$ and $z'=x'+iy'$, use these to obtain the transformation equations in terms of the rotation matrix for *vector rotation.* **(2 Marks)**

(b) Consider two vectors: $|1\rangle=\begin{pmatrix}1\\2\end{pmatrix}$ and $|2\rangle=\begin{pmatrix}-1\\2\end{pmatrix}$

i. What is the angle between the two vectors $|1\rangle$ and $|2\rangle$? **(1 Mark)**

ii. use the rotation matrix you derived above for *vector rotation* to rotate the given vectors in real space by an angle of 45° (on your answer sheet plot the vectors before rotation and after rotation) **(1 Mark)**

iii. If $|1'\rangle$ and $|2'\rangle$ are the rotated vectors that you computed above what is the angle between these vectors? **(1 Mark)**

2. Given two equations with two unknowns:

$$a_1X+b_1Y=c_1$$
$$a_2X+b_2Y=c_2$$

Cramer's rule says that the solution to this equation (provided the matrix $\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}$ is non-singular) is given as: $X=\dfrac{\begin{vmatrix}c_1 & b_1\\ c_2 & b_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$ and $Y=\dfrac{\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$. For *rotation of axes*, the transformation equations are given as:

$$x' = \cos\theta\, x + \sin\theta\, y$$
$$y' = -\sin\theta\, x + \cos\theta\, y$$

Use Cramer's rule defined above to find x and y in terms of x' and y'.

(5 Marks)

3. Gram-Schmidt orthogonalization is recursively performed for a given basis vectors set $\{|v_k\rangle\}$. To begin with, $|u_1\rangle = |v_1\rangle$; subsequently in index notation, the normalized set $\{|u_i\rangle\}$ is found by: $|u_j\rangle = |v_j\rangle - \sum_{i;i\neq j}^{j-1} \frac{\langle v_j|u_i\rangle}{\langle u_i|u_i\rangle}|u_i\rangle$. Perform the Gram-Schmidt

   Orthogonalization on the basis: $|v_1\rangle = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $|v_2\rangle = \begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $|v_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$. **(5 Marks)**

4. For the matrix $M = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$, find the

   (a) eigenvalues **(2 Marks)**

   (b) in the special case where $h = \sqrt{2}(a-b)$ obtain the eigenvectors. **(3 Marks)**

5. Perform the summations (or obtain the expressions) indicated in the following index expressions (all indexes vary from 1 to 3)

   (a) $\epsilon_{ijk} v_i v_j v_k$; how many terms are there in this sum? **(2 Marks)**

   (b) $u_i v_j \delta_{ji} - v_m v_n \delta mn$ **(1 Mark)**

6. Consider two vectors: $|1\rangle = \begin{pmatrix}1\\1\end{pmatrix}$ and $|2\rangle = \begin{pmatrix}-1\\1\end{pmatrix}$

   (a) Normalize the two vectors to obtain $|\hat{1}\rangle$ and $|\hat{2}\rangle$ **(1 Mark)**

   (b) Consider a transformation described by $M = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$ described in the natural basis.

   Transform M into the bases described by the normalized vectors: $|\hat{1}\rangle$ and $|\hat{2}\rangle$

   (4 Marks)

7. Using Gram-Schmidt process in the domain $[0,\infty)$ for the function set $\{u_j(x)|j=0,1,2\} = \{1, x, x^2\}$ with a weight function $\omega(x) = e^{-x}$, obtain the first three Laguerre polynomials $L_0(x)=1$ $L_1(x)=x-1$ and $L_2(x)=\frac{1}{2}(x^2-4x+2)$. The Gram-Schmidt procedure following the identifications mentioned above, involves: (1) find $\psi_j(x) = u_j(x) - \sum_{i;i\neq j}^{j-1} a_{ji} u_i(x)$ (2) substitute $a_{ji} = -\int_a^b u_j \varphi_i \omega\, dx$ and (3) normalise $\psi_j(x)$ to obtain $\varphi_j(x)$ i.e. $\varphi_j(x) = \frac{\psi_j(x)}{\sqrt{\int_a^b \psi_j^2 \omega\, dx}}$ where $\omega(x)$ is a weight function. In this particular problem, you may need the following integral (for the Gamma function): $\int_0^\infty x^n e^{-x} dx = n!$ **(5 Marks)**

8.

(a) A particular similarity transformation yields

$$A' = U A U^{-1}$$
$$A^{\dagger\prime} = U A^{\dagger} U^{-1}$$

If the adjoint relationship is preserved i.e. ($A^{\dagger\prime} = A'^{\dagger}$) and the determinant of U is 1 i.e. $\det U = 1$, show that U is Unitary **(3 Marks)**

(b) Using index notation show that $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$ **(2 Marks)**

9.

(a) Complex numbers, $a+ib$, with a and b real, may be represented by 2x2 matrices:

$a+ib \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that this matrix representation is valid for

i. addition of two complex numbers **(1 Mark)**

ii. multiplication of two complex numbers **(1 Mark)**

(b) Find the matrix corresponding to $(a+ib)^{-1}$ **(3 Marks)**

10.

(a) Find the real and imaginary parts of the following as functions of x and y

i. $w = 2z$ **(1 Mark)**

ii. $w = z^{*}$ **(1 Mark)**

iii. $w = z^2 + 2z + i$ **(1 Mark)**

(b)

i. Express $w = z^2 + z + 1$ in the form of $u + iv$ **(1 Mark)**

ii. Show that u and v satisfy the Cauchy-Riemann conditions. **(1 Mark)**