Registration Number:
Date & session:

ST.JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU -27

M.Sc (PHYSICS) – III SEMESTER
SEMESTER EXAMINATION: OCTOBER 2022
(Examination conducted in December 2022)
PH 9120 – QUANTUM MECHANICS-II

Time: 2 ½ Hours **Max Marks: 70**

This paper contains 5 printed pages and 2 parts

PART A

Answer any FIVE full questions. **(5x10=50)**

1.
(a) Define Parity Operator. Prove that the parity operator is Hermitian and unitary.

(b) Explain how symmetric and antisymmetric wave functions are constructed from unsymmetrized solutions of the Schrodinger equation of a system of indistinguishable particles. [5+5]

2.

(a) A particle of mass m is moving in a one-dimensional box defined by the potential $V(x)=\begin{cases}0 & 0\le x\le a\\ \infty & \text{Otherwise}\end{cases}$. Estimate the ground state energy using the trial function: $\psi(x)=Ax(a-x),\quad 0\le x\le a$.

(b) In the WKB approximation, assuming the momentum $p(x)=\sqrt{2m(E-V(x))}$ to be varying slowly and assuming the wavefunction to be of the form: $\psi(x)=A(x)e^{i\phi(x)}$, obtain the differential equations for $A(x)$ and $\phi(x)$. [5+5]

3. For a Hydrogen atom placed in a constant electric field ε , obtain an expression for the perturbed Hamiltonian. Using this, derive the first order perturbation to the ground state Energy. The wavefunction for the ground state of Hydrogen atom is: $\psi_{100}(r,\theta,\phi)=\frac{1}{\sqrt{\pi a^3}}e^{-r/a}$.

4. For time dependent perturbation on a state $|\phi_i\rangle$, the components of eigenstates evolve through the equation: $i\hbar \frac{d b_k^{(1)}}{dt} = e^{i\omega_{ki} t}\langle \phi_k|\hat{W}|\phi_i\rangle$ to put the system into the final state $|\phi_k\rangle$.

   (a) Write this as an integral equation.

   (b) From the integral equation for b_k write down the transition probability P_{if} from an initial state i to a final state f. [2+8]

5. A two fold degenerate system ($|\phi_n^{(0)}\rangle = c_l|\psi_l^{(0)}\rangle + c_k|\psi_k^{(0)}\rangle$) is perturbed by a Hamiltonian $W = \lambda \hat{W}$ where λ is a parameter that characterizes the perturbation such that $\lambda \ll 1$ and the new Hamiltonian is given as $H = H_0 + \lambda \hat{W}$ (where H_0 is the unperturbed Hamiltonian such that $H_0|\psi_l^{(0)}\rangle = E_l^{(0)}|\psi_l^{(0)}\rangle$ and $H_0|\psi_k^{(0)}\rangle = E_l^{(0)}|\psi_k^{(0)}\rangle$). If we assume that the eigenfunctions as well as the energy in the Schrodinger equation: $H|\psi_n\rangle = E_n|\psi_n\rangle$ are also perturbed such that $|\psi_n\rangle = |\phi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \ldots$ and $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots$. Obtain and expression for $E_n^{(1)}$.

6. Write down the time-independent Schrodinger Equation *of a Free Particle* in 3-dimensions. If the wavefunction is assumed to be separable, decompose the system into the radial, polar and azimuthal equations. You may take the Laplacian in Spherical Polar Coordinates to be described by: $\nabla^2 = \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial \phi^2}\right]$.

7. Describe scattering by a potential with a figure. Explain the various terms and obtain the asymptotic form for the scattered wave.

PART-B

Answer any 4 questions **(4x5=20)**

[Constants: **h**=6.626070x10^{-34} J s (**Planck's constant**), **1eV** = 1.6x10^{-19} J (**electron volt to Joules**), **c**=2.99792458x10^{8} m/s (**speed of light**), **1Å** = 1x10^{-10}m (**Angstrom to meters**), **e** = 1.602176x10^{-19} C (**electronic charge**), **ε_0**= 8.85418782x10^{-12}m^{-3}kg^{-1}s^{4}A^{2} (**permittivity of free space**), **m_{proton}**=1.672621898x10^{-27}kg (**mass of proton**), **$m_{electron}$**=9.10938356x10^{-31}kg (**mass of electron**), **$m_{neutron}$**=1.674927471x10^{-27}kg (**mass of neutron**), **a** = 5.029x10^{-10}m (**Bohr radius**), **α** = 1/137 (**Fine Structure Constant**), **G**=6.674x10^{-11}m^{3}kg^{-1}s^{-2} (**Gravitational constant**), **$M_\odot$**=1.9891x10^{30} kg (**Solar mass**), **$R_\odot$**=6.9x10^{8} m (**Sun's Radius**), **σ** = 5.67x10^{-8}

8. The matrices of the unperturbed ($H^{(0)}$) and the perturbation (W) Hamiltonians in the orthonormal basis $|\phi_1\rangle$ and $|\phi_2\rangle$ are: $H^{(0)}=\begin{pmatrix} E_0+\epsilon & 0 \\ 0 & E_0-\epsilon \end{pmatrix}$, $W=\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$.

Determine

(a) The first order correction to energy

(b) Second order correction to energy

(c) Wave function corrected to first order [1+2+2]

9. Given that the transition probability from state i to state f is given by the expression $P_{if}(t)=\frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_{fi}t'}\langle\phi_f|W|\phi_i\rangle dt'\right|^2$ where $\omega_{fi}=\frac{E_f-E_i}{\hbar}$ and W is the *time dependent perturbation* to the Hamiltonian. Compute the transition from ground state for a Harmonic Oscillator subjected to a perturbation $W=-x e^{-t^2/t_0^2}$ from $t=0$ to $t=\infty$ (use the table of integrals given).

10. Assuming a bouncing ball (bouncing from the ground to maximum height) to be having an extra bounce such that the combined gravitational and bounce potential is described by a potential of the form: $V(y)=ky^2$ where k is a constant and y represents the vertical coordinate along which the ball is bouncing. If H represents that maximum height to which the ball bounces, determine the energy levels of the bouncing ball. The connection formula implies that $\psi(y)=\frac{2F}{\sqrt{p(y)}}\sin\left(\frac{1}{\hbar}\int_y^H p(y')dy'+\frac{\pi}{4}\right)$ for $y<H$.

11. Normalize the trial function $e^{\alpha r}$ and evaluate the ground state energy of the hydrogen atom (given $V(r)=\frac{ke^2}{r}$ and use ∇^2 in Spherical Polar Coordinates).

12. Give the zeroth- order wave functions for helium atom

(a) in the ground state $1s^2$

(b) in the excited state $1s^1 2s^1$.

13. A particle scatters off of a finite barrier step potential containing three distinct regions with the total energy $E>0$, $E<V_0$ and $E>0$ respectively, where V_0 is the height of the barrier. Obtain the relationship of the incident amplitude with that of the transmitted amplitude.

Table of (some) Integrals

Gamma Function:
$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}\,dt$
$\Gamma(n)=(n-1)!$
$\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!}{4^n n!}\sqrt{\pi}$

(a) $\int_0^{\infty} e^{-2bt}\,dt=\frac{1}{2b}$

(b) $\int_0^{\infty} t\,e^{-2bt}\,dt=\frac{1}{4b^2}$

(c) $\int_0^{\infty} t^2 e^{-2bt}\,dt=\frac{1}{4b^3}$

(d) $\int_0^{\infty} t^3 e^{-2bt}\,dt=\frac{3}{8b^4}$

(e) $\int_0^{\infty} t^4 e^{-2bt}\,dt=\frac{3}{4b^5}$

(f) $\int_0^{\infty} t^5 e^{-2bt}\,dt=\frac{15}{8b^6}$

(g) $\int_0^{\infty} t^6 e^{-2bt}\,dt=\frac{45}{8b^7}$

(h) $\int \frac{1}{t^2+b^2}\,dt=\frac{1}{b}\tan^{-1}\left(\frac{t}{b}\right)$

(i) $\int \frac{1}{(t^2+b^2)^2}\,dt=\frac{1}{2b^3}\left(\frac{bt}{(b^2+t^2)}+\tan^{-1}\left(\frac{t}{b}\right)\right)$

(j) $\int_0^{\infty} t^4 e^{-\alpha^2 t^2}\,dt=\frac{3\sqrt{\pi}}{8\alpha^5}$

(k) $\int \frac{1}{(t^2+b^2)^4}\,dt=\frac{1}{16b^7}\left(\frac{15t^5 b+40b^3 5^3+33b^5 t}{(3t^6+9bt^4+9b^3t^2+b^5)}+5\tan^{-1}\left(\frac{t}{b}\right)\right)$

(l) $\int \frac{t^2}{(t^2+b^2)^2}\,dt=\left(-\frac{t}{(2b^2+2t^2)}+\frac{1}{2b}\tan^{-1}\left(\frac{t}{b}\right)\right)$

(m) $\int \frac{1}{(t^2+b^2)^3}\,dt=\frac{3}{8b^5}\left(\frac{5/3\,b^3 t+bt^3}{(b^2+t^2)^2}+\tan^{-1}\left(\frac{t}{b}\right)\right)$

(n) $\int \frac{t^2}{(t^2+b^2)^4}\,dt=\frac{1}{16b^5}\left(\frac{bt^5+8/3\,b^3t^3-b^5t}{(b^2+t^2)^3}+\tan^{-1}\left(\frac{t}{b}\right)\right)$

(o) $\int \frac{t^4}{(t^2+b^2)^4}\,dt=\frac{1}{16b^3}\left(\frac{bt^5+8/3\,b^3t^3-b^5t}{(b^2+t^2)^3}+\tan^{-1}\left(\frac{t}{b}\right)\right)$

(p) $\int \frac{t^6}{(t^2+b^2)^4}\,dt=\frac{1}{16b}\left(\frac{11bt^5+40/3\,b^3t^3-5b^5t}{(b^2+t^2)^3}+5\tan^{-1}\left(\frac{t}{b}\right)\right)$

(q) $\int \sqrt{a/x-1}\,dx=x\sqrt{a/x-1}+a\ \tan^{-1}\left(\sqrt{a/x-1}\right)$

(r) $\int \sqrt{1-ax}\,dx=-\frac{2(1-ax)^{3/2}}{3a}$

(s) $\int \sqrt{1-ax^2}\,dx=\frac{1}{2}x\sqrt{1-ax^2}+\frac{\sin^{-1}\sqrt{a}\,x}{2\sqrt{a}}$

(t) $\int_{-\infty}^{\infty} e^{-\alpha t^2+i\omega t}\,dt=\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{\omega^2}{4\alpha}}$

(u) $\int_0^{\infty} t^n e^{-st}\,dt=\frac{n!}{s^{n+1}}$ (Laplace Transform)

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J J ... M M ...
m_1 m_2	
m_1 m_2	Coefficients

1/2×1/2

m_1	m_2	1, +1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

m_1	m_2	5/2, +5/2
+2	+1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

m_1	m_2	3, +3
+3/2	+3/2	1

m_1	m_2	3, +2	2, +2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3, +1	2, +1	1, +1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3, 0	2, 0	1, 0	0, 0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3, −1	2, −1	1, −1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3, −2	2, −2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3, −3
−3/2	−3/2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2	7/2, +7/2
+2	+3/2	1

m_1	m_2	7/2, +5/2	5/2, +5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2, −5/2	5/2, −5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2, −7/2
−2	−3/2	1

2×2

m_1	m_2	4, +4
+2	+2	1

m_1	m_2	4, +3	3, +3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4, +2	3, +2	2, +2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4, +1	3, +1	2, +1	1, +1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4, −1	3, −1	2, −1	1, −1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4, −2	3, −2	2, −2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4, −3	3, −3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4, −4
−2	−2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.