Registration Number:

Date & Session:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27

B.Sc. Mathematics - V SEMESTER

SEMESTER EXAMINATION: OCTOBER 2023

(Examination conducted in November/December 2023)

MT 5123 – MATHEMATICS - V

(For current batch students only)

Time: 2 hrs **Max Marks: 60**

This paper contains TWO printed pages and THREE parts.

Part A

Answer any SIX of the following. **6 X 2 = 12**

1. In a ring R, prove that (i) $a \cdot 0 = 0, \forall a \in R$ (ii) $a(-b) = -(ab)$, $\forall\, a, b \in R$.
2. Find all the idempotent elements of an integral domain R.
3. Define the characteristic of a ring. What is the characteristic of the ring $\mathbb{Z}_4 \oplus \mathbb{Z}_6$.
4. Show that $2\mathbb{Z} = \{2x \mid x \in \mathbb{Z}\}$ is an ideal of the ring of integers $\mathbb{Z}$.
5. Define the kernel of a ring homomorphism. Find the kernel of the ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}_4$ defined by $f(x) = x(\text{mod } 4)$.
6. Show that the necessary condition for the integral $\int_{x_1}^{x_2} (y^2 + (y')^2 + 2ye^x)\, dx$ to be extremum is $y'' - y = e^x$.
7. What are the geodesics on (i) a plane and (ii) a sphere?
8. If the length of the curve joining the points $A(a, \theta_1, \varphi_1)$ and $B(a, \theta_2, \varphi_2)$ on a sphere of radius a is given by $L = \int_{\theta_1}^{\theta_2} a\, \sqrt{1 + (\varphi')^2 \sin^2 \theta}\; d\theta$, show that the necessary condition for L to be minimum is $\dfrac{\varphi' \sin^2 \theta}{\sqrt{1 + (\varphi')^2 \sin^2 \theta}} = c$, where *c* is a constant.

Part B

Answer any FIVE of the following. **5 X 6 = 30**

9. Define a ring. Find the zero element, the unity, and the units of the ring $(\mathbb{Z},\oplus,\otimes)$, where $a\oplus b=a+b+1,\ a\otimes b=a+b+ab,\ \forall a,b\in\mathbb{Z}$.

10. Prove that a nonempty subset S of a ring R is a subring of R if and only if $\forall a,b\in S$, $a-b\in S$ and $ab\in S$. Hence show that $aR=\{ar\mid r\in R\}$, for some $a\in R$, is a subring of R.

11. Define an integral domain. Prove that there is no integral domain of order 6. Give the reason why there are no fields with 6 elements.

12. Define a prime ideal. Prove that an ideal I of a commutative ring R with unity is a prime ideal if and only if R/I is an integral domain.

13. Prove that for any integer $n>1$, the ideal $n\mathbb{Z}=\{na\mid a\in\mathbb{Z}\}$ of the ring $\mathbb{Z}$ is a maximal ideal if and only if n is a prime.

14. Show that the mapping $f:\mathbb{Z}_6\to\mathbb{Z}_{30}$, defined by $f(x)=10x(\bmod 30)$, is a ring homomorphism. Find $\ker(f)$.

15. If $\varphi:R\to S$ is a ring homomorphism, prove that $R/\ker(\varphi)\cong\varphi(R)$. [The first isomorphism theorem].

Part C

Answer any THREE of the following. **3 X 6 = 18**

16. Show that the necessary condition for the integral $I=\int_{x_1}^{x_2}F(x,y,y')\,dx$ to be an extremum is that $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$.

17. (i) Deduce the Euler's equation for a functional $F(x,y,y')$ that does not contain independent variable x explicitly.

    (ii) Find the extremal of the integral $\int_{x_1}^{x_2}[y+(y')^2]\,dx$. **(4+2)**

18. Show that the geodesic on a right circular cylinder is a circular helix.

19. If a cable hangs freely under the action of gravity from two fixed points, show that it hangs in the form of a catenary.

20. Find the extremal of the functional $\int_0^1[x+(y')^2]\,dx$ subject to the constraint $\int_0^1 y\,dx=1$, given $y(0)=0$ and $y(1)=1$.