Register Number:

Date and session:

ST JOSEPH'S UNIVERSITY, BENGALURU - 27
M.Sc (MATHEMATICS) - I SEMESTER
SEMESTER EXAMINATION: OCTOBER 2023
(Examination conducted in November/December 2023)
MT7221: REAL ANALYSIS
(For current Batch students only)

Duration: 2 Hours **Max. Marks:** 50

---

1. The paper contains **TWO** printed pages and **ONE** part.
2. Attempt any **FIVE FULL** questions.

---

1. a) Show that every superset of an infinite set is an infinite set.
   b) Prove that union of two denumerable sets is denumerable. **[6+4]**

2. a) Let $f:[a,b]\longrightarrow\mathbb{R}$ be bounded on $[a,b]$. Let P be a partition of $[a,b]$ and Q be a refinement of P. Show that $U(P,f)\geq U(Q,f)$ and $L(P,f)\leq L(Q,f)$.
   b) A function f is defined on $[0,1]$ by
   $$f(x)=\begin{cases}x, x\in[0,1]\cap\mathbb{Q}\\ 0, x\in[0,1]-\mathbb{Q}\end{cases}$$
   Find the upper and lower integral of f. **[6+4]**

3. a) Let $f:[a,b]\longrightarrow\mathbb{R}$ be bounded on $[a,b]$ and f be continuous on $[a,b]$ except for a finite number of points in $[a,b]$. Prove that f is integrable on $[a,b]$.
   b) State and prove the fundamental theorem of integral calculus. **[5+5]**

OR

a) Let $f:[a,b]\to\mathbb{R}$ be integrable and for each $x\in[a,b]$ define $f(x)=\displaystyle\int_a^x f(t)dt$. Show that F is differentiable at any point $c\in[a,b]$ at which f is continuous and that $F'(c)=f(c)$.

b) Define pointwise convergence of a sequence of functions (f_n) defined on a domain $\mathbb{D}\subseteq\mathbb{R}$. Determine the set of points on which the sequence (f_n) is pointwise convergent where $f_n:\mathbb{R}\longrightarrow\mathbb{R}$ is defined by $f_n(x)=x^n$. **[6+4]**

4. a) For the series $\sum_{n=1}^{\infty} f_n(x)$ where $f_n(x) = n^2 x e^{-n^2x^2} - (n-1)^2 x e^{-(n-1)^2x^2}, x \in [0,1]$, show that $\sum_{n=1}^{\infty}\left(\int_0^1 f_n(x)dx\right) \neq \int_0^1 \left(\sum_1^{\infty} f_n(x)\right) dx$.

   b) A sequence of functions (f_n) is defined on $[0,1]$ by $f_n(x) = \dfrac{nx}{1+n^2x^2}, x \in [0,1]$. Show that the sequence (f_n) is not uniformly convergent on $[0,1]$. **[5+5]**

5. a) Prove that a finite set in a metric space is always closed.

   b) Show that the mapping $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0,\infty)$ is metric on $\mathbb{R}^2$ where $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$, $x = (x_1,x_2)$ and $y = (y_1,y_2) \in \mathbb{R}^2$. **[5+5]**

6. a) Show that the sequence space ℓ^{∞} is complete for $1 \leq p \leq \infty$.

   b) Show that a set S in a metric space X is bounded iff there exists $x_0 \in X$ and $r > 0$ such that $S \subseteq B(x_0, r)$. **[6+4]**

7. a) Let $f : (X, d_X) \longrightarrow (Y, d_Y)$ be a mapping between metric spaces. Show that f is continuous iff the inverse $f^{-1}(V)$ of each d_Y-open set V of Y is a d_X-open subset of X.

   b) Let $f : X \longrightarrow Y$ be a function. Show that f is continuous iff for every subset $B \subseteq Y, f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$. **[5+5]**

************* **BEST WISHES** *************