Register Number:

Date: 5-1-21

St. Joseph's College (Autonomous), Bangalore
M.Sc Mathematics-I Semester
Semester Examination: January 2021
MT7118: Algebra-I

Duration: 2.5 Hours **Max. Marks:**70

1. The paper contains **THREE** printed pages.
2. Answer any **SEVEN FULL** questions, where each question carries 10 marks.
3. The last part of each question is an MSQ which may have one or more correct option.
4. In this paper $|g|$ denotes the order of an element g in a group G.
5. Any ring mentioned here is commutative and has unity unless otherwise mentioned.

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1. (a) Prove that the subgroup A_n consisting of all even permutations is normal S_n . **[5 marks]**
   (b) In $D_{10} = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$, find $|r^3|$ and $|sr^2|$. **[2 marks]**
   (c) S_6 has **[3 marks]**
   i) 180 elements of order 4.
   ii) 60 elements of order 2.
   iii) 90 elements of order 4.
   iv) 75 elements of order 2.

2. (a) If G is a finite group of order n and p is the smallest prime dividing order of the group, then prove that any subgroup of index p is normal. **[7 marks]**
   (b) Which of the following is/are true in $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ **[3 marks]**
   i) $Z(D_8) = \{1\}$
   ii) $Z(D_8) = \{1, r^2\}$
   iii) $cl(r^2) = \{r^2\}$
   iv) $cl(1) = \{1\}$

3. (a) State the Third Isomorphism Theorem for group. **[2 marks]**
   (b) Let G is a group and $H \trianglelefteq G$. Show that for any $g \in G$ the map $\sigma_g : H \to H$ defined by $\sigma_g(h) = ghg^{-1}$ is an automorphism of H. **[5 marks]**
   (c) Which of the following is/are true? **[3 marks]**

i) $|\text{Aut}(\mathbb{Z}_5)| = 4$

ii) $|\text{Aut}(\mathbb{Z}_8)| = 8$

iii) $|\text{Aut}(\mathbb{Z}_{10})| = 4$

iv) $|\text{Aut}(\mathbb{Z}_7)| = 4$

4. (a) Prove that if G is a group of order 132, then G is not simple. **[7 marks]**

   (b) Which of the following is/are true? **[3 marks]**

   i) A group of order 200 is simple.

   ii) A group of order 55 is simple.

   iii) A group of order 1000 is not simple.

   iv) A group of order 40 is not simple.

5. (a) Find the distinct conjugacy classes of S_3. **[4 marks]**

   (b) State the Division algorithm for F[x], where F is a field. Show that $f(a)$ is the remainder of $f(x)$ when divided by $(x-a)$ in $F[x]$. **[4 marks]**

   (c) The remainder of x^{2021} when divided by $x-4$ in $\mathbb{Z}_5$ is/are **[2 marks]**

   (i) 1 (ii) 4 (iii) 0 (iv) -1

6. (a) State and prove Eisenstein's Criterion. **[7 marks]**

   (b) Determine which of the following is/are irreducible over indicated rings **[3 marks]**

   (i) $x^5 - 3x^4 + 2x^3 - 5x + 8$ over $\mathbb{R}$.

   (ii) $x^3 + 2x^2 + x + 1$ over $\mathbb{Q}$.

   (iii) $x^3 + 3x^2 - 6x + 3$ over $\mathbb{Z}$.

   (iv) $x^4 + x^2 + 1$ over $\mathbb{Z}_2$.

7. (a) Let F be a field. Show that if $p(x) \in F[x]$ is irreducible then $\langle p(x) \rangle$ is a maximal ideal in $F[x]$. **[4 marks]**

   (b) Construct a field with 25 elements. **[3 marks]**

   (c) Let $I_1 = \langle x^2 + 1 \rangle$ and $I_2 = \langle x^3 - x^2 + x - 1 \rangle$ in $\mathbb{Q}[x]$. If $R_1 = \mathbb{Q}[x]/I_1$ and $R_1 = \mathbb{Q}[x]/I_2$ then which of the following is/are true? **[3 marks]**

   (i) R_1 and R_2 are both fields.

   (ii) R_1 is a field but R_2 is not a fields.

   (iii) R_1 is an integral domain but R_2 is not an integral domain.

   (iv) R_1 and R_2 are both integral domains.

8. (a) Show that $\mathbb{Z}[i]$ is a Euclidean Domain. **[4 marks]**

   (b) Show that in a Unique Factorization Domain every irreducible element is prime. **[4 marks]**

   (c) Which of the following is/are true? **[2 marks]**

   (i) A subring of an integral domain is an integral domain.

   (ii) A subring of a Unique factorization domain(UFD) is a UFD.

   (iii) A subring of a Principal ideal domain(PID) is a PID.

   (iv) A subring of a Euclidean domain is a Euclidean domain.

9. (a) Prove that every non-zero prime ideal in a Principal Ideal Domain is a maximal ideal. **[4 marks]**

   (b) Let R be a commutative ring such that $R[x]$ is a Principal Ideal Domain. Show that R is a field. **[3 marks]**

   (c) Which of the following statement(s) is/are true? **[3 marks]**

   (i) $\mathbb{Z}[x]$ is a PID.

   (ii) $\mathbb{Z}[x, y]/\langle y + 1\rangle$ is a unique factorization domain.

   (iii) If R is a prinicipal ideal domain and p is a non-zero prime ideal of R, then R/p has finitely many prime ideals.

   (iv) If R is a principal ideal domain then any subring of R containing the unity 1 is again a principal ideal domain.

10. (a) Let F be a field and $f(x)$ be a non-constant polynomial in $F[x]$. Prove that there is a field E containing an isomorphic copy of F in which $f(x)$ has a root. **[5 marks]**

   (b) Let K/F be a field extension with $[K : F] = 2$ and $\text{char}(F) \neq 2$. Show that $K = F(\sqrt{D})$ where D is not a square element in K. **[3 marks]**

   (c) Which of the following is/are true? **[2 marks]**

   (i) $\mathbb{Q}(\sqrt{2} + \sqrt[3]{2}) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$

   (ii) $x^3 - 2$ is irreducible over $Q(\sqrt{2})$

   (iii) $[\mathbb{Q}(\sqrt{2} + \sqrt[3]{2}) : \mathbb{Q}] = 4$

   (iv) $[\mathbb{Q}(\sqrt{2} + \sqrt[3]{2}) : \mathbb{Q}(\sqrt{2})] = 2$

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