Register Number:

Date: 07-01-2021

St. Joseph's College (Autonomous), Bangalore
M.Sc Mathematics - I Semester
End Semester Examination: January, 2021
MT7218: Real Analysis

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains three pages.
2. Attempt any **SEVEN FULL** questions.
3. All multiple choice questions have **one or more** correct option. Write **all** the correct options in your answer booklet.

1. a) Let A be an arbitrary set. Show that the cardinality of A is less than the cardinality of its power set $\mathcal{P}(A)$, that is $|A| < |\mathcal{P}(A)|$. **[4m]**

   b) Show that $\mathbb{N} \times \mathbb{N}$ is denumerable. **[3m]**

   c) Which of the following are countable? **[3m]**

      A. The set of all functions from the set of natural numbers $\mathbb{N}$ to $\{0,1\}$.

      B. The set of all functions from the set $\{0,1\}$ to natural numbers $\mathbb{N}$.

      C. The set of all finite subsets of $\mathbb{N}$.

      D. The set of all subsets of $\mathbb{N}$.

2. a) Let (X,d) be a metric space. Show that $d_1 : X \times X \to \mathbb{R}$ defined by $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is also a metric on X. **[7m]**

   b) Let (X,d) be a metric space. Which of the following is/are also a metric on X? **[3m]**

      A. $\dfrac{d(x,y)}{1-d(x,y)}$ B. $1 - \dfrac{1}{1+d(x,y)}$ C. $\dfrac{1}{1-d(x,y)} - 1$ D. $d(x,y)\cdot(1-d(x,y))$

3. a) Define open sphere and open set in a metric space. Show that in a metric space (X,d) every open sphere is an open set. **[4m]**

   b) Prove that a finite union of closed sets is closed. **[3m]**

   c) Let A be a subset of $\mathbb{R}$ where $A \neq \emptyset, \mathbb{R}$. Which of the following is/are true? **[3m]**

      A. If A is closed then $A = \overline{(A^\circ)}$. B. If A is open then $A = (\overline{A})^\circ$.

      C. If A is open then $A = \left(\overline{(A)^c}\right)^c$. D. If A is closed then $A = \left((A^c)^\circ\right)^c$.

4. a) State and prove the **converse** of the Cantor's intersection theorem. **[7m]**

   b) Write whether the statements below are true or false. No justification required. **[3m]**

      A. If $F_1 \supseteq F_2 \supseteq \cdots$ are distinct, non-empty subsets of a metric space then $\bigcap_{i=1}^{n} F_i \neq \emptyset$ always.

      B. If $F_1 \supseteq F_2 \supseteq \cdots$ are distinct, non-empty closed and bounded subsets of a metric space then

$\bigcap_{i=1}^{n} F_i$ has a single point.
C. If $F_1 \supseteq F_2 \supseteq \cdots$ are distinct, non-empty closed and bounded subsets of $\mathbb{R}$ then $\bigcap_{i=1}^{n} F_i$ has a single point.

5. a) Prove that a monotone function on $[a, b]$ is Riemann integrable. **[5m]**
   b) Let $f \in \mathcal{R}[a, b]$ and suppose $f[a, b] \subseteq [c, d]$ and $\phi : [c, d] \to \mathbb{R}$ be a continuous function. Prove that $\phi \circ f \in \mathcal{R}[a, b]$. **[3m]**
   c) Pick out the true statement(s): **[3m]**
      A. Every bounded function is integrable.
      B. Every integrable function is bounded.
      C. If $\int_a^b f = 0$ then $f = 0$.
      D. If f is continuous and $\int_a^b f = 0$ then $f = 0$.

6. a) State and prove the second form of the Fundamental Theorem of Calculus **[7m]**
   b) The value of $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f'\left(\frac{k}{3n}\right)$ is **[3m]**
      A. $3[f(1/3) - f(0)]$
      B. $f(1/3) - f(0)$
      C. $\frac{1}{3}[f'(3) - f'(0)]$
      D. $\int_0^1 f'\left(\frac{x}{3}\right) dx$

7. a) Let f be a Darboux integrable function on $[a, b]$. Show that f is Riemann integrable on $[a, b]$. **[4m]**
   b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function at all points $x_0 \in \mathbb{R}^n$. Show that the directional derivative of f exists at x_0 in the direction v for any $v \in \mathbb{R}^n$ and that $D_v f(x_0) = f'(x_0)v$ **[4m]**
   c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $f(x, y) = (x + y, x - y)$. Which of the following is/are true? **[2m]**
      A. f is differentiable everywhere.
      B. f is differentiable only at $(0, 0)$
      C. f is differentiable at $(0, 0)$ and $f'(0, 0) = f$
      D. $f'(x, y) = f$ for all $(x, y) \in \mathbb{R}^2$

8. a) State and prove the mean value theorem for a multivariable differentiable function. **[5m]**
   b) Compute the partial derivatives and hence the matrix that corresponds to the derivative of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = xyz$ at the point $(1, 1, 1)$. **[2m]**
   c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be the map $f(x_1, x_2, \cdots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $a = (a_1, \cdots, a_n)$ is a fixed non-zero vector. Let $Df(0)$ denote the derivative of f at 0. Which of the following is/are ture? **[3m]**
      A. $[Df(0)](a) = \|a\|^2$
      B. $[Df(0)]$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$.
      C. $[Df(0)]a = 0$.
      D. For any $b = (b_1, \cdots, b_n)$, $[Df(0](b) = a_1b_1 + \cdots + a_nb_n$.

9. a) Let (f_n) be a sequence of uniformly continuous functions on $[a, b]$.. Show that if f_n uniformly converges to $f : [a, b] \to \mathbb{R}$ then f is uniformly continuous. **[3m]**
   b) Show that the sequence $f_n(x) = x - \frac{x^n}{n}$ converges uniformly on $[0, 1]$ but that the derivative sequence (f_n') does not. **[5m]**
   c) Which of the following sequences $\{f_n\}$ is uniformly convergent on $[0, 1]$? **[2m]**
      A. $f_n(x) = x^n$
      B. $f_n(x) = 1 - \frac{x}{n}$
      C. $f_n(x) = \frac{x^n}{n}$
      D. $f_n(x) = 1 - x^n$

10. a) Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers. Show that $\sum_{n=1}^{\infty} \frac{a_n x^n}{1+x^{2n}}$ is uniformly and absolutely convergent for all $x \in \mathbb{R}$. **[5m]**

b) Show that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is uniformly convergent on $\mathbb{R}$. **[2m]**

c) Which of the following is true for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}(2x-1)^n$ **[3m]**

A. The series converges on $(0,1]$
B. The series converges on $(0,1)$
C. The radius of convergence is 1
D.The radius of convergence is $1/2$

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