Register Number:

Date: 23-11-2020

St. Joseph's College (Autonomous), Bangalore
M.Sc Mathematics - III Semester
End Semester Examination: November, 2020

MTDE9518: Algebraic Topology

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains two pages.
2. Answer any **SEVEN FULL** questions.
3. All multiple choice questions have 1 or more than one correct option. Full marks will be awarded only for writing **all correct options** in your answer script.

1. a) Show that a contractible space is path connected. Also show that $\mathbb{R}^n$ is a contractible space for any $n \geq 1$. **[4+2m]**

   b) A subset A of $\mathbb{R}^n$ is star convex if there is a point $a \in A$ such that for any $x \in A$, the straight line joining x to a is completely in A. Pick out which among the following is/are star convex sets. **[4m]**

   i. Any contractible subset of $\mathbb{R}^n$. ii. $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$

   iii. Any convex subset of $\mathbb{R}^n$. iv. $\{(x,y) \in \mathbb{R}^2 : xy = 1\}$.

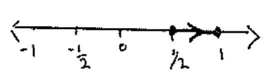
2. a) Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X based at x_0. Show that the operation "$*$" of concatenation is associative on $\pi_1(X, x_0)$. **[6m]**

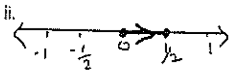
   b) Given spaces X and Y, let $[X, Y]$ denote the set of **homotopy classes** of maps from X to Y. Let $Y = S^2 - \{(0,0,1), (0,0,-1)\}$. Then the number of elemnets in $[\mathbb{R}, Y]$ is: **[4m]**

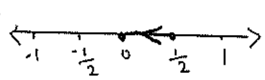
   i. 1 ii. 2 iii. Uncountably many iv. Countably many.

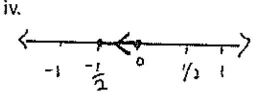
3. a) Prove that the fundamental group of the circle is isomorphic to the additive group of integers. **[7m]**

   b) Let $p : \mathbb{R} \to S^1$ be the map $p(t) = (\cos 2\pi i t, \sin 2\pi i t)$. Which of the following is a lift of the path $f(s) = (-\cos(\pi s), \sin(\pi s))$ via p? **[3m]**

i.


ii.


iii.


iv.


4. a) Show that a covering map is an open map. **[6m]**

b) Let $p : E \to B$ be a covering map. Which of the following are true? **[4m]**
i. If B is connected then so is E.
ii. If B is simply connected then so E.
iii. If E is simply connected then so is B.
iv. If B is simply connected and E is path connected then E is simply connected.

5. a) Show that the quotient map $q : S^2 \to \mathbb{RP}^2$ is a covering map. **[7m]**

b) Which of the followig are covering maps? **[3m]**
i. $p : S^1 \to S^1$ given by $p(x) = -x^2$.
ii. $p : (0, \infty) \to S^1$ given by $p(t) = e^{2\pi i \log t}$.
iii. $p : S^2 \to S^2$ given by $p(x, y, z) = (-x, -y, -z)$.

6. a) Let $h : S^1 \to X$ be a continuous function. Show that if h is nulhomotopic then h extends to a continuous map $k : B^2 \to X$. **[7m]**

b) Which of the following are nulhomotopic? **[3m]**
i. The map $h : S^1 \to S^1$ given by $h(x) = -x$.
ii. The projection map $h : S^1 \to [-1, 1]$ given by $h(x_1, x_2) = x_1$.
iii. The inclusion map $j : S^1 \to \mathbb{R}^2$.

7. a) Let $f : (X, x_0) \to (Y, y_0)$ be continuous. If f is a homotopy equivalence then show that $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. **[7m]**

b) Which of the following are homotopy equivalent to S^1? **[3m]**
i. $S^1 \times I$ ii. $S^2 - \{(0, 0, 1)\}$ iii. $S^2 - \{(0, 0, 1), (0, 0, -1)\}$ iv. $\mathbb{R}^3 - \{0\}$

8. a) Show that any continuous function $f : B^2 \to B^2$ has a fixed point. Further, if A is a retract of B^2 then show that any continuous function $f : A \to A$ has a fixed point. **[7m]**

b) Let $f : [-1, 1] \to [-1, 1]$ be a function satisfying $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$. Then the number of fixed points of f is: **[3m]**
i. 0 ii. 1 iii. infinite

9. a) Let $p : E \to B$ be a covering map. Let e_0 and e_1 be points of $p^{-1}(b_0)$ and let $H_i = p_*(\pi_1(E, e_i))$. Prove that H_0 and H_1 are conjugates, that is, if γ is a path in E from e_0 to e_1 and $\alpha = p \circ \gamma$ then show that $[\alpha] * H_1 * [\alpha]^{-1} = H_0$. Conversely, let $e_0 \in E$ and let H be a subgroup of $\pi_1(B, b_0)$ conjugate to H_0, then prove that there exists a point $e_1 \in p^{-1}(b_0)$ such that $H = H_1$. **[6m]**

b) Let $p : E \to B$ and $p' : E' \to B$ be covering maps. Let $p(e_0) = p'(e'_0) = b_0$. Show that the covering maps p and p' are equivalent if and only if the subgroups $H_0 = p_*(\pi_1(E, e_0))$ and $H'_0 = p'_*(\pi_1(E'e'_0))$ of $\pi_1(B, b_0)$ are conjugate. **[4m]**

10. a) Suppose $X = U \cup V$, where U and V are open subsets of X. Suppose that $U \cap V$ is path connected and $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the respective incusion maps. Show that the images of the induced homomorphisms $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$, generate $\pi_1(X, x_0)$. **[8m]**

b) Using the Van-Kampen theorem, compute the fundamental group of the Θ space. **[2m]**

MTDE9518_A 20