Register Number:

Date:

ST. JOSEPH'S COLLEGE (AUTONOMOUS), BENGALURU-27
M.SC MATHEMATICS - III SEMESTER
SEMESTER EXAMINATION: OCTOBER, 2021
(Examination conducted in January-March 2022)
MTDE 9318: COMMUTATIVE ALGEBRA

Duration: 2.5 Hours **Max. Marks:** 70

1. The paper contains two pages.
2. Attempt any **SEVEN FULL** questions.
3. Each question carries 10 marks.

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1. Let A be a ring and let

$$A[[x]] = \{f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \mid a_i \in A\}$$

be the ring of formal power series with coefficients in A.

(a) Show that the contraction of a maximal ideal $\mathfrak{m}$ of $A[[x]]$ is a maximal ideal of A. [**6 marks**]

(b) Moreover, show that $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and x. [**3 marks**]

(c) In general, is the contraction of a maximal ideal always maximal? Justify. [**1 mark**]

2. (a) State and prove prime avoidance lemma. [**7 marks**]

(b) Show that a local ring has only trivial idempotent elements. [**3 marks**]

3. (a) Let A be a ring and $\text{Nil}(A)$ its nilradical. Show that the following are equivalent:

(i) A has exactly one prime ideal.

(ii) Every element of A is either a unit or a nilpotent.

(iii) $A/\text{Nil}(A)$ is a field. [**5 marks**]

(b) Show that $\left(\frac{\mathbb{Q}}{\mathbb{Z}}\right) \otimes_{\mathbb{Z}} \left(\frac{\mathbb{Q}}{\mathbb{Z}}\right) = 0.$ [**3 marks**]

(c) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic as $\mathbb{R}$-modules. **[2 marks]**

4. (a) Let M be a finitely generated A- module and S be a multiplicatively closed subset of A. Show that $S^{-1}(\text{Ann}(M)) = \text{Ann}(S^{-1}M)$. [**7 marks**]

(b) Show by an example that the condition "M is a finitely generated A-module" in 4(a) is necessary. [**3 marks**]

5. Let A be a nonzero ring and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A. [**10 marks**]

6. Prove 'Lying over theorem': "Let B/A be an integral extension. Every prime ideal of A is contraction of a prime ideal of B. Further, two distinct prime ideals q and q' of B such that $q \subset q'$ cannot contract to the same prime ideal of A. [**10 marks**]

7. Let M be a module having a composition series of length n. Show that any composition series of M has length n. Moreover, show that any decreasing chain of submodules can be extended to a composition series of M. [**10 marks**]

8. Show that in a Noetherian ring every ideal has a primary decomposition by proving the following results,

   (a) In a Noetherian ring every ideal is a finite intersection of irreducible ideals. [**4 marks**]

   (b) In a Noetherian ring every irreducible ideal is primary. [**6 marks**]

9. State and prove Hilbert Basis theorem. [**10 marks**]

10. (a) Prove that in an Artin ring every prime ideal is maximal. [**4 marks**]

    (b) Prove that an Artin ring has only finitely many maximal ideals. [**6 marks**]

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